Coulomb Gas Representation of Low Energy QCD

Sebastian Jaimungal and Ariel R. Zhitnitsky
Department of Physics and Astronomy, University of British Columbia, Vancouver, BC V6T 1Z1, Canada
(February 1, 2008)

A novel Coulomb gas (CG) description of low energy QCD_4, based on the dual transformation of the QCD effective chiral Lagrangian, is constructed. The CG is found to contain several species of charges, one of which is fractionally charged and can be interpreted as instanton-quarks. The creation operator which inserts a pseudo-particle in the CG picture is explicitly constructed and demonstrated to have a non-zero vacuum expectation value. The Wilson loop operator as well as the creation operator for the domain wall in the CG representation is also constructed.

arXiv:hep-ph/9904377v3 22 Jun 2000

1. Color confinement, spontaneous breaking of chiral symmetry, the $U(1)$ problem, θ dependence, and the classification of vacuum states are some of the most interesting questions in QCD. Unfortunately, the progress in our understanding of them is extremely slow. At the end of the 1970s A. M. Polyakov [1] demonstrated color confinement in QED_3; this was the first example in which nontrivial dynamics was a key ingredient in the solution. Soon after, 't Hooft and Mandelstam [2] suggested a qualitative picture of how confinement could occur in QCD_4. The key point, the 't Hooft - Mandelstam approach, is the assumption that dynamical monopoles exist and Bose condense. Many papers have been written on this subject since the original formulation [2]; however, the main questions, such as, "What are these monopoles?"; "How do they appear in the gauge theories without Higgs fields?"; "How do they interact?", "What is the relation (if any) between confinement and instantons?" were still not understood. Almost 20 years passed before the next important piece of the puzzle was solved [3]. Seiberg and Witten demonstrated that confinement occurs in SUSY QCD_4 due to the condensation of monopoles much along the lines suggested many years ago by 't Hooft and Mandelstam (for a recent review see [4]). Furthermore, condensation of dyons together with oblique confinement for nonzero vacuum angle, θ, was also discovered in SUSY models (a phenomenon which was also argued to take place in ordinary QCD; see [4]). In addition to forming concrete realizations of earlier ideas, the recent progress in SUSY models has introduced many new phenomena, such as the existence of domain walls [5] which connect two distinct vacua with the same θ.

In this letter we shall illuminate one of the missing elements mentioned above by demonstrating that the dual representation of the low-energy effective chiral Lagrangian corresponds to a statistical system of interacting pseudo-particles with fractional $1/N_c$ charges. We shall identify these particles with instanton quarks suspected long ago [6], [7], consequently demonstrating a link between confinement and instantons.

Through a rewriting of the low energy effective QCD_4 action [8], in terms of a dual Coulomb gas (CG) picture, the operator which creates pseudo-particles will be demonstrated to have a non-zero vacuum expectation value (vev). Many additional interesting features will arise in the dual CG picture:

i) there are several species of pseudo-particles which interact according to the Coulomb law;

ii) the species conjugate to the flavor singlet component of the chiral condensate will be shown to correspond to fractionally charged pseudo-particles;

iii) the operator which creates the pseudo-particles in the dual CG picture will be constructed in terms of the phases of chiral condensate;

iv) the Wilson loop operator insertion in the dual CG picture will also be constructed.

Our analysis begins with the effective low energy QCD action derived in [8], which allows the θ-dependence of the ground state to be analyzed. Within this approach, the Goldstone fields are described by the unitary matrix U_{ij}, which correspond to the γ_5 phases of the chiral condensate: $\langle \overline{\Psi}_L^i \Psi_R^j\rangle = -|\langle \overline{\Psi}_L \Psi_R\rangle| \, U_{ij}$ with $U = \exp\left[i\sqrt{2}\,\frac{\pi^a\lambda^a}{f_\pi} + i\frac{2}{\sqrt{N_f}}\frac{\eta'}{f_{\eta'}}\right]$, $UU^+ = 1$, where λ^a are the Gell-Mann matrices of $SU(N_f)$, π^a is the pseudo-scalar octet, and $f_\pi = 133\ MeV$. In terms of U, the low-energy effective potential is given by [8]:

$$W_{QCD} = -\lim_{V\to\infty} \frac{1}{V} \ln \sum_{r=-\infty}^{\infty} \sum_{l=0}^{p-1} \exp\Big\{ VE\cos\left(-\frac{q}{p}\theta + i\frac{q}{p}\log \mathrm{Det}\, U + \frac{2\pi}{p}\, l - 2\pi r\right) + \frac{1}{2}V\,\mathrm{Tr}\,(mU + m^+U^+)\Big\}, \quad (1)$$

where V is the volume of the system. All dimensional parameters in this potential are expressed in terms of the QCD vacuum condensates, and are well known numerically: $m = \mathrm{diag}(m_q^i|\langle\overline{\Psi}^i\Psi^i\rangle|)$; and the constant E is related to the QCD gluon condensate $E = \langle b\alpha_s/(32\pi)G^2\rangle$. The only unknown parameters in this construction are the integers p, q, which play the same role as the discrete integer numbers classifying the vacuum states in SUSY theories. The only constraint on them is that in large N_c limit, $q/p \sim 1/N_c$ such that the $U(1)$ problem is resolved.

To convince the reader that (1) does indeed represent the anomalous effective Lagrangian, three of its most salient features are listed below (for details see [8]):

i) Eq.. (1) correctly reproduces the VVW effective chiral Lagrangian [9] in the large N_c limit;

ii) it reproduces the anomalous conformal and chiral Ward identities of QCD;

iii) it reproduces the known dependence in θ for small angles [9]; however, it may lead to different behavior for large values $\theta > \pi/q$ if $q \neq 1$. Accordingly, it leads to the correct 2π periodicity of observables.

From this point onwards $U = \exp\{i\mathrm{diag}(\phi_1, \dots, \phi_{N_f})\}$, are the phases of the chiral condensate and $\phi = \mathrm{Tr}\ \ln U$ represents the singlet field*. The effective potential then takes on a specific Sine-Gordon (SG) form. Such a structure is quite natural for terms proportional to m_i and is associated with the Goldstone origin of the ϕ_i fields. A similar Sine-Gordon structure for the singlet combination is less obvious and corresponds to the following behavior of the $(2k)^{\mathrm{th}}$ derivative of the vacuum energy in pure gluodynamics,

$$\frac{\partial^{2k} E_{vac}(\theta)}{\partial\, \theta^{2k}}\Big|_{\theta=0} \sim \int \prod_{i=1}^{2k} dx_i \langle Q(x_1) ... Q(x_{2k}) \rangle \sim (\frac{i}{N_c})^{2k},$$

where $Q \sim G_{\mu\nu}\widetilde{G}_{\mu\nu}$. This property was seen as a consequence of the solution of the $U(1)$ problem [10].

2. Although there are several interacting fields in the effective QCD action (1) many of the special properties of the SG theory apply to this model, and the admittance of a CG representation [1] for the partition function is no different. The existence of many fields and cosine terms only serve to make the formulae more bulky while the basic strategy remains the same. Using the effective potential (1) the defining partition function is taken to be†,

$$Z = \sum_{k=0}^{p-1} \sum_{n=-\infty}^{\infty} \int \mathcal{D}\phi_a \ \mathrm{e}^{-g \int d^4x (\vec{\nabla}\phi_a)^2} \tag{2}$$
$$\times\ e^{\int d^4x \left\{ E \cos\left(\frac{q}{p}(\phi - \theta + 2\pi n) + 2\pi k\right) + \sum_{a=1}^{N_f} m_a \cos(\phi_a) \right\}}$$

where the coupling constant $g \propto f_\pi^2$. Performing a series expansion in E and m_a, and introducing the $\mathbb{Z}_2$ valued fields, $Q^{(a)}$ for $a = 0, \dots, N_f$, to represent the cosine interactions, makes it possible to rewrite the partition function in a form in which the dynamical scalar fields can be integrated out exactly. Carrying out the integration then leads to the novel dual CG picture with action,

$$\begin{aligned} S_{CG} = i\theta Q_T^{(0)} - \ln\left(p \sum_{n=-\infty}^{\infty} \mathrm{e}^{2\pi i Q_T^{(0)} n}\right) \\ + \frac{1}{2g^2} \sum_{a=1}^{N_f} \Bigg\{ \sum_{b,c=1}^{M_0} Q_b^{(0)}\ G(x_b^{(0)} - x_c^{(0)})\ Q_c^{(0)} \\ + 2 \sum_{b=1}^{M_0} \sum_{c=1}^{M_a} Q_b^{(0)}\ G(x_b^{(0)} - x_c^{(a)})\ Q_c^{(a)} \\ + \sum_{b,c=1}^{M_a} Q_b^{(a)}\ G(x_b^{(a)} - x_c^{(a)})\ Q_c^{(a)} \Bigg\}. \end{aligned} \tag{3}$$

The species $Q_i^{(0)}$ are dual to the singlet field $\phi(x)$, while the species $Q_i^{(a\neq 0)}$ are dual to the phases, $\phi^a(x)$, of the chiral condensates $\langle \overline{\Psi}^a \Psi^a \rangle$. Also, $Q_T^{(0)} = \sum_{b=1}^{M_0} Q_b^{(0)}$ is the total $Q^{(0)}$ charge for that configuration and $G(x-y)$ denotes the relevant Greens function of the Laplace operator. The full partition function is,

$$Z = \sum_{M_0, \dots, M_{N_f}=0}^{\infty} \sum_{\substack{Q_i^{(0)} = \pm\frac{q}{p} \\ Q_i^{(a\neq 0)} = \pm 1}} \frac{(\frac{E}{2})^{M_0}}{M_0!} \frac{(\frac{m_1}{2})^{M_1}}{M_1!} \dots \frac{(\frac{m_{N_f}}{2})^{M_{N_f}}}{M_{N_f}!}$$
$$\times \int (dx_1^{(0)} \dots dx_{M_0}^{(0)}) \dots (dx_1^{(N_f)} \dots dx_{M_{N_f}}^{(N_f)})\ \mathrm{e}^{-S_{CG}}. \tag{4}$$

Notice that the fugacities of the species-$(a \neq 0)$ are given by the masses of the a^{th} quark, while the fugacity of the species-(0) is proportional to the gluon condensate E. An important point is that the fugacity of the species-$(a \neq 0)$ vanishes in the chiral limit, while that of the (0) species remains non-zero. A second point is that, in the chiral limit, it is not obvious whether (4) is independent of θ, while in QCD the θ-angle appears with quark masses and hence disappears in this limit. It is possible to demonstrate that the partition function has this property; however, the details are deferred to an extended version of this letter [11].

3. There are several important features of the action (3) which should be noted. Firstly, the summation over n forces the total $Q^{(0)}$ charge, $Q_T^{(0)}$, to be an integer. Such a constraint is the analog of the quantization of the topological charge and is to be expected. Secondly, due to the manner in which $Q^{(0)}$ appears with the parameter

---

* It is well known that this is the most general form for U-matrix describing the ground state of the system [9]. The off-diagonal elements of U describe the fluctuations of the physical Goldstones which are neglected. This is allowed since only the diagonal elements are relevant in the description of the ground state which is the focus of this work. Furthermore, such a truncation can be justified a posteriori by demonstrating that the classification of vacuum states based on the CG representation exactly coincides with the classification based on the effective Lagrangian approach, see Eq. (1) where only the diagonal elements are relevant. Finally, as will be demonstrated shortly, the most important contribution is related to the *singlet field*, $\phi = \mathrm{Tr}\ \ln U$, which is unambiguously defined. In fact, all contributions related to the non-singlet fields are suppressed in the chiral limit by the quark masses m_i, and can be neglected as a first approximation.

†The following fact was used to replace the summations over r and l in (1) with k and n in (2): given the pair of integers $(r \in \mathbb{Z}, l = 0, ..., p-1)$ there exists a unique pair of integers $(n \in \mathbb{Z}, k = 0, ..., p-1)$, such that: $\frac{q}{p}n - k = \frac{l}{p} - r$

θ and the fact that its total charge, $Q_T^{(0)}$, is integer, one can identify $Q_T^{(0)}$ as the total topological charge (defined in four dimensional Euclidean space) of the given configuration. Indeed, in QCD the θ parameter appears in the Lagrangian only in the combination $i\theta\frac{G_{\mu\nu}\widetilde{G}_{\mu\nu}}{32\pi^2}$. It is quite interesting that although the starting effective low-energy Lagrangian is colorless, we have ended up with a representation in which a colorfull object (the topological charge $Q_T^{(0)}$) can be identified. Furthermore, since the (0) species has charges $\sim \frac{q}{p} \sim 1/N_c$, the integer constraint enforces a fractional quantization on the pseudo-particles: the difference in the number of positively and negatively charged pseudo-particles of species-(0) must be an integer multiple of p. This simple observation implies that our fractionally charged pseudo-particles, $Q^{(0)}$, cannot be related to any semi-classical solutions, which can carry only integer charges; rather, configurations with fractional charges should have pure quantum origin.

Let us reiterate the reasons for this identification: The fact that species $Q_i^{(0)}$ has charges $\sim 1/N_c$ is a direct consequence of the θ/N_c dependence in the underlying QCD with frozen, non-dynamical quarks; in addition, due to the 2π periodicity of the theory, only configurations which contain an integer topological number contribute to the partition function. Therefore, the number of particles $Q_i^{(0)}$ with charges $\sim 1/N_c$ must be proportional to N_c.

Note that the θ-angle only appears with $Q_T^{(0)}$, this is a direct consequence of the pseudo-particles of species-(0) being dual to the singlet field ϕ which is the only field appearing with θ in the effective low-energy action (1). A final point about the form of the CG action is that the θ-dependence supplies an overall phase factor for each configuration and leads to the very natural interpretation of non-trivial θ-angles as introducing an overall background charge. Turning attention to the interactions amongst the pseudo-particles: species-(0) is seen to interact with all species-$(a = 0, \ldots, N_f)$; however, all other species, $(a \neq 0)$, only interact with their own species and species-(0). This peculiar behave is, once again, due to species-(0)'s association with the singlet while the other species are associated with particular components of the chiral condensate.

In order to attach a physical meaning to each charge $Q_i^{(0)}$, as opposed to the total topological charge $Q_T^{(0)}$, we would like to remind the reader about an interesting connection between the CG statistical ensemble and the $2d$ $O(3)$ σ-model (more generally, CP^{N_c} models) [6]. An exact accounting and resummation of the n-instanton solutions maps the original problem to a $2d$-CG with fractional charges (the so-called instanton-quarks). These pseudo-particles, instanton-quarks, do not exist separately as individual objects; rather, they appear in the system all together as a set of $\sim N_c$ instanton-quarks so that the total topological charge of each configuration is always integer. This means that a charge for an individual instanton quark cannot be created and measured; instead, only the total topological charge for the whole configuration is forced to be integer and has a physical meaning. However, a convenient parameterization of the instantons can be made by associating a charge of strength $\sim 1/N_c$ to the pseudo-particles. In fact, such an interpretation in terms of these (fictitious) fractionally charged objects leads to an elegant explanation of confinement and other important properties of the model [6]. With this in mind, the discussion is now returned to the CG system (3). Notice that if $q = 1$ and $p = N_c$ (as arguments based on SUSY suggest [5]), then, the number of integrations over $d^4x_i^{(0)}$ in Eq.(4) exactly equals $4N_ck$, where k is integer. This is a consequence of the fact that the number of species-(0) pseudo-particles must be an integer multiple of N_c, see discussion after Eq.(4). This number, $4N_ck$, exactly corresponds to the number of zero modes in the k-instanton background, and we *conjecture* that (at low energies[‡]) the fractionally charged species-(0) pseudo-particles are the instanton-quarks suspected long ago [7]. An interesting point is that if the gauge group, G, was not $SU(N)$, the number of integrations would be equal to $4kC_2(G)$ where $C_2(G)$ is the quadratic Casimir of the gauge group. This is because the θ dependence in physical observables will then appear in the combination $\frac{\theta}{C_2(G)}$, and all appearances of N_c are replaced by $C_2(G)$. This number $4kC_2(G)$ exactly corresponds to the number of zero modes in the k-instanton background for gauge group; hence, this analysis correctly reproduces the scenario for arbitrary gauge groups, and supplies further support for our conjecture.

4. The relationship between a CG representation for the classical gas of pseudo-particles and the effective quantum field theory for a scalar field ϕ is not a new idea and has been known since [1]. The general strategy in this approach is to express all observables (Wilson loop, etc.) in terms of the gas of pseudo-particles and use the effective description in terms of the auxiliary ϕ field simply for convenience of calculations. However, it has proven to be useful [13] to consider the ϕ field as a main player and to obtain the operator in the ϕ description which introduces a pseudo-particle at the point x in the CG (this operator will be denoted $\mathcal{M}$). The analysis [13] of the corresponding correlation functions in the Polyakov model of the operator $\mathcal{M}$ leads to the same explanation of confinement which was already known. However, new insights about confinement extending from such analysis (e.g. condensation being a consequence of $\langle\mathcal{M}\rangle \neq 0$) are found. In particular, the ϕ field has the physical meaning of a potential (indeed, the action den-

[‡]The analysis carried out here is based upon a low-energy effective action, as such nothing can be said about the high energy behavior of our dual model.

sity of the system is proportional to $-\cos(\phi)$) describing a statistical ensemble of pseudo-particles in the mean-field approximation. The condensate $\langle\mathcal{M}\rangle \sim e^{i\langle\phi\rangle} \neq 0$ was named "magnetization", and the ϕ field was called the scalar magnetic potential due to the fact that pseudo-particles in 3d are magnetic monopoles.

We would like to carry out a similar analysis, and construct the operator, $\mathcal{M}$, which creates a pseudo-particle (instanton quark) at the point x. It can be demonstrated [11] that the operator which inserts a single pseudo-particle of species-(a) with charge, q_a, in the bulk of the gas can be written in terms of the phases of the chiral condensate as follows:

$$\mathcal{M}(\mathbf{q}_a,\mathbf{X}) = \begin{cases} \mathrm{e}^{i\mathbf{q}_0\left(\phi(\mathbf{X})-\theta+2\pi n+2\pi\frac{p}{q}k\right)} & , \quad a=0 \\ \mathrm{e}^{i\mathbf{q}_a\phi_a(\mathbf{X})} & , \quad a\neq 0 \end{cases} \quad (5)$$

Such an exponential form for the creation operator $\mathcal{M}$ in terms of the scalar ϕ field is quite similar to what was found for Polyakov's model [13]. This should not be surprising, since such a form is a universal property for CG representations and does not depend on the dimension of the space-time.

With equal, small, quark masses, $m \to 0$, the vev's in the semi-classical limit are $\langle\phi_a\rangle \sim \frac{\theta}{N_f}$ for small θ [8] implying that in this regime, $\langle\mathcal{M}\rangle \neq 0$. Recall that our starting point, the effective action, was constructed under the assumption that the system is in the confining phase, i.e. (1) contains only colorless degrees of freedom. This confinement (which was already implemented into our system) is described in terms of the ϕ_a fields by a non-zero vev $\langle\mathcal{M}\rangle \sim \mathrm{e}^{i\langle\mathbf{q}_a\phi_a\rangle} \neq 0$. Therefore, it is tempting to assume that the property $\langle\mathcal{M}\rangle \neq 0$ is intimately related to confinement in any dimension as was checked in 2d CP^{N-1} model [6], 3d Polyakov's model [13] and 4d QCD(5).

Due to the existence of non-trivial solutions to the classical equations of motion of the effective action [14], an operator in the CG representation which induces a source for those solutions should exist. Indeed, it is not difficult to show [11] that the operator is,

$$\mathcal{C}(\eta_a) \propto \mathrm{e}^{i\sum_{b=1}^{M_0} Q_b^{(0)}\eta(x_b^{(0)})} \prod_{a=1}^{N_f} \mathrm{e}^{i\sum_{b=1}^{M_a} Q_b^{(a)}\eta_a(x_b^{(a)})}. \quad (6)$$

where, $\eta \equiv \eta_1 + \ldots + \eta_{N_f}$. This operator inserts sources, $\int d^4x\phi_a\Box\eta_a$, into the effective action (2). Consequently, fluctuations about a classical field configuration can be taken into account in the CG picture by introducing a phase factor which gives rise to a source term for that background - all interactions among the charges remain unaltered. An interesting source is given by $\eta(x) = \frac{2\pi}{q}\mathrm{sgn}(x_0)$ which, in the SG representation, gives rise to the domain wall solutions that interpolate between vacuum states labeled by k and $k+1$ [14]. Such a source term in the CG clearly interacts non-trivially with *all* charges. Similar domain walls are known to exist in supersymmetric models (see [5] for a review).

As a final important identification, consider an insertion of the source

$$\Box\eta = 4\pi\theta_S(x_1,x_2)\int dy_\lambda\epsilon^{\lambda\sigma}\partial_\sigma\delta^2(x-y), \ \lambda,\sigma = 0,3, \quad (7)$$

where $\theta_S(x_1,x_2)$ is unity within a surface S and zero outside. One can show [11] that it corresponds to the insertion of a Wilson loop operator in the (x_1,x_2)-plane. In the SG picture (2) the Wilson loop insertion corresponds to the shift $\phi \to \phi+\eta$ in the flavor-singlet potential term $\sim E\cos(\frac{q}{p}(\phi+\eta-\theta+2\pi n)+2\pi k))$.

5. To conclude: the existence of the free parameter θ plays the role of a messenger between colorless and colorful degrees of freedom. The CG picture developed has a rich structure in which several species of pseudo-particles appear and the unique fractionally charged species is found to be explicitly affected by the presence of a θ-angle.

As a final concluding remark, at the intuitive level there seems to be a close relation between out CG representation in terms of instanton quarks and the "periodic instanton" analysis [15]. Indeed, in [15] has been shown that the large size instantons and monopoles are intimately connected and, therefore, instantons have the internal structure. Unfortunately, one should not expect to be able to account for large instantons using semi-classical technique to bring this intuitive correspondense onto the quantitative level.

Besides that, the recent analysis [16], where it was demonstrated that in SYM the instanton quarks carry magntetic charges and saturate the gluino condensate, also supports our picture. Finally, the recent analysis [17]of dynamical symmetry breaking in $SU(N_c)$ and $USp(2N_c)$ gauge theories, shows the condensation of " magnetic quarks" with fractional charges rather than magnetic monopoles itself. This phenomenon again supports our picture where we claim that the instanton constituents (instanton quarks) are appropriate degrees of freedom in description of low energy QCD.

Discussions on dense instanton ensemble and monopoles with Pierre van Baal, T. Schafer and E. Shuryak are greatly appreciated.

[1] A. M. Polyakov, Nucl. Phys. **B120** (1977) 429.

[2] G't Hooft, in "Recent Developments in Gauge Theories" Charges 1979, Plenum Press, NY 1980; Nucl. Phys. **B190** (1981) 455; S. Mandelstam, *Phys. Rep.* **23** (1976) 245.

[3] N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994) 19; Nucl. Phys. **B431** (1994) 484.

[4] G.'t Hooft, in Confinement, Duality and Nonperturbative Aspects of *QCD*, P. van Baal, Ed, NATO ASI Series, 1998 Plenum Press, NY 1998; G.'t Hooft, hep-ph/9812204.

[5] M. Shifman, Progr. Part. Nucl. Phys. **39** (1997) 1.
[6] V.Fateev et al, Nucl. Phys. **B154** (1979) 1; B.Berg and M.Luscher, Commun.Math.Phys. **69**(1979) 57.
[7] A. Belavin et al, Phys. Lett. **83B** (1979) 317.
[8] I. Halperin and A.R. Zhitnitsky,*Phys. Rev. Lett.* **81**, 4071 (1998); *Phys. Rev.* **D58**, 054016 (1998), hep-ph/9711398.
[9] E. Witten, Ann. Phys. **128** (1980) 363; P. Di Vecchia and G. Veneziano, Nucl. Phys. **B171** (1980) 253.
[10] G. Veneziano, Nucl. Phys. **B159** (1979) 213.
[11] S. Jaimungal and A.R. Zhitnitsky, hep-ph/9905540.
[12] E. Witten, Phys. Lett. **86B** (1979) 283.
[13] N. Snyderman , Nucl. Phys. **B218** (1983) 381.
[14] T. Fugleberg, I. Halperin and A.R. Zhitnitsky, hep-ph/9808469, *Phys. Rev.* **D59** 074023 (1999).
[15] T.C. Kraan and P. van Baal, Nucl. Phys. **B533** (1998) 627; Phys. Lett.**B435** (1998) 389; M.G.Perez, T.G.Kovacs, P. van Baal Phys. Lett.**B472** (2000) 295
[16] N. M. Davies et al, hep-th/9905015.
[17] G.Carlino, K. Konishi and H. Murayama, hep-th/0001036, hep-th/0005076.